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THE COSMOLOGICAL MASS DISTRIBUTION. II. DISORDERED TREES AND THE ADHESION APPROXIMATION

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ABSTRACT

The mass distribution of cosmic structures has been connected in Paper I to the number of random trajectories (overdensities as a function of mass scale) in a Cayley tree. These may diffuse and/or branch with decreasing mass, and the two processes were related to the Press & Schechter mass distribution or to the Smoluchowski aggregation equation, respectively.

Here we start from the *dynamical* equations for the cosmic matter field, specifically in the adhesion approximation, and we show that the distribution of the masses condensed into isotropic "schocks" is given by a specific instance of the above branching diffusion process, with the branching rate provided by the adhesion approximation. Such equivalence establishes a *bridging* between the dynamics of the cosmic fluid under gravity and the phenomenological mass distributions generated (as limiting cases) by our model based on branching diffusion.

We discuss how this approach can be extended beyond the adhesion approximation, to describe the emergence of cosmic structure as a random process of the branching diffusion kind.

Subject headings: cosmology: theory — galaxies: formation — large-scale structure of universe — methods: numerical

1. INTRODUCTION

One of the main goals in the theory of cosmic structure formation is to predict the mass distribution $N(M, t)dM$ of the high-constrast structures and its evolution with the cosmic time t. Here we concentrate on the range of masses $10^{12}\ M_\odot \lesssim M \lesssim 10^{15}\ M_\odot$ corresponding to groups and clusters of galaxies. Such masses are sufficiently large to play down the role of dissipation, and sufficiently small to yield today large contrasts and high signals in the observations.

The canonical approach, pioneered by Peebles (1965) and discussed by Doroshkevich (1970), is based on the assumption that structures form hierarchically by gravitational instability of overdense regions in the "initial" density field, assumed to constitute at redshifts $10^3 \lesssim z \lesssim 10^4$ a Gaussian random process. In this vein, Press & Schechter (1974, hereafter PS) proposed their classic formula for $N(M, t)$. The derivation has been discussed by many authors (including Schaeffer & Silk 1988; Peacock & Heavens 1990; Bower 1991) also considering the topology of the density field (see Bardeen et al. 1986).

More recently, the approach has been reformulated interestingly in terms of random walks by Bond et al. (1991). Given the initial density field under the form of the contrasts $\delta \equiv \Delta\rho/\rho \ll 1$ over the background density ρ, one envisages it being filtered over progressively smaller scales; one then looks for the regions in which δ in its linear development rises above a threshold $\delta_c \sim 1$, taken to ensure the region condensation. Bond et al. argued that, in the simplest case of a filter sharp in the Fourier k-space, the density contrasts execute a *simple*, Markovian random walk as the mass scale decreases. Correspondingly, the overdensities δ diffuse away from zero with the decrease of the scale. The rate by which they upcross the threshold δ_c yields the mass fraction condensed on that scale. The result is the PS formula.

However, it is widely recognized (see Peacock & Heavens 1990; Bond et al. 1991; Bond & Myers 1993; Lacey & Cole 1993; Bertschinger & Jain 1994; Monaco 1995) that such a formula, although yielding surprisingly good fits to N-body simulations with increasing dynamic ranges and resolutions (provided δ_c is treated as a tunable parameter) rests on shaky foundations and on ad hoc assumptions: the perturbations are taken to evolve over a homogeneous isotropic background; they do so independently on all scales, with no interactions; the complex nonlinear dynamics is reduced to the extrapolation of the linear behavior up to the threshold δ_c; mass assignment and value of δ_c are taken from one specific perturbation shape, not fully consistent with the first and the second assumption; in particular, the canonical value $\delta_c \approx 1.69$ refers to the evolution of spherical, homogeneous, and isolated perturbations up to the contrast ~ 180, which ought to mark full virialization.

On the other hand, there is evidence that binary aggregation processes play a considerable role in the formation of astrophysical structures; cases in point are provided by the buildup of a cD-like galaxy in a group, or even by the marks left by aggregations on morphology, dynamics, and abundance of bright ellipticals (see Bhavsar 1989; Caon et al. 1994). In addition, very recent observations (Dressler et al. 1994; Glazebrook et al. 1995; Windhorst et al. 1995) suggest that interactions of sub-L_* galaxies play a role increasing with z and provide evidence that aggregations and merging may contribute considerably to the observed "excess" of faint, blue galaxy counts.

Such processes are well described in terms of an aggregation kinetic equation for $N(M, t)$ of the Smoluchowski kind (Cavaliere, Colafrancesco, & Menci 1992). The latter authors stressed that aggregations ought to affect the evolution of other substructures inside larger condensations, like subgroups in galaxy clusters and groups or clusters in superclusters. Specific observational evidence is increasing (see Schindler 1995). It is now widely accepted (see Doroshkevich et al. 1995; Sahni & Coles 1995; Shandarin 1995) that the cellular large-scale structure plays an active role in guiding and promoting aggregations between smaller knots in a stage subsequent to their formation.

This second kind of kinetics corresponds to a stochastic process *different* from a simple, diffusive random walk, since aggregations are not straightforwardly related to the statistics of the initial δ field, and they proceed through random binary coalescence of condensations. In the conventional direction from large to small scales, successive aggregations may be referred to as a cascade of random "branchings" of one condensation into two; see Figure 1.

How aggregations are combined with, or nested into, the collapse of overdensities constitutes a challenging issue. Cavaliere & Menci (1994, hereafter Paper I) proposed a *model* which unifies the two underlying stochastic processes into a single one, of the class termed "branching with disorder" (see Derrida & Spohn 1988), or "branching diffusion" (see Feller 1966; Karlin & Taylor 1975), hereafter BD. This may be visualized (see Fig. 1) in terms of a Cayley tree characterized by a partition function which Paper I relates to the mass distribution. The PS formula and the Smoluchowski equation are then recovered as limiting cases.

The model had been proposed on a phenomenological basis. Diffusion (describing the collapse of overdense regions) and branching (describing the aggregation of condensations) were coupled by the branching rate η, a parameter not predicted within the model.

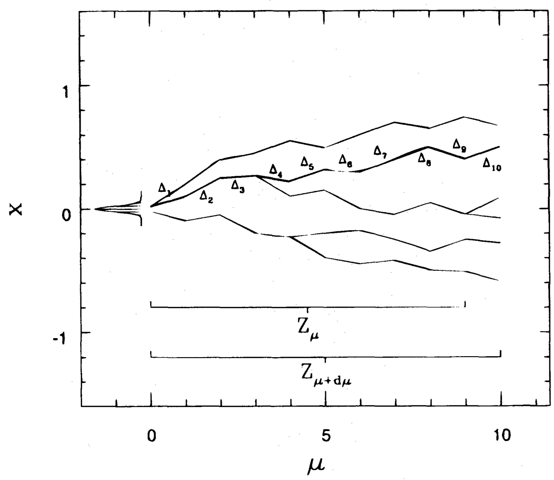

FIG. 1.—A schematic representation of a disordered tree with branching. Random weights Δ_i are extracted (from a Gaussian) on the links i of each tree path j. The paths may branch into two. As a consequence, the quantity $x + x_j$ (see text) fluctuates with increasing step number $\mu \propto M^{-2a}$. A particular path is marked by the heavy line, and the corresponding random weights are indicated explicitly. The distribution in x of the trajectories $Z(x, \mu)$ follows the recursion relation (2.6). Its initial Gaussian shape is sketched on the left. At increasing μ, the distribution deviates increasingly from a Gaussian, as described by equation (2.7).

In the present paper, we discuss how the BD process can be connected with the *dynamics* of the cosmic fluid under gravity, governed in full by the Euler, continuity, and Poisson equations. Perhaps the most effective way devised so far to handle them is provided by the adhesion approximation (AA; Gurbatov, Saichev, & Shandarin 1989), which allows for an explicit solution in terms of the velocity and opens the way to neat geometrical representations. We start from the statistics of "shocks" which form in such an approximation, as developed recently in the limit of small masses by Sinai (1992a) and by Vergassola et al. (1994).

We intend to show that such statistics is equivalent to that governing the BD process. Thus, we provide a *dynamical* foundation for the model in Paper I. The ability of the latter to yield the PS formula or the aggregation kinetics in the appropriate limits now appears to arise from a complex, nonlinear dynamics including direct collapses, *and* aggregations of condensations to some extent. So the PS formula finds its natural place, and the role of binary aggregations can be assessed.

The plan of the paper is as follows: in § 2 we recall the diffusive stochastic approach to the mass function introduced by Bond et al. (1991), and we reformulate the BD model of Paper I in terms appropriate to the present purpose. In § 3 we make contact with the results for the statistics of shocks in the AA derived by Sinai (1992a) and extended by Vergassola et al. (1994). In § 4 we show that such statistics is equivalent to a BD process in the small mass range, and we derive the specific value of the branching rate provided by AA. That such equivalence holds also at large masses is shown in § 5. In § 6 we show that the BD representation includes the aggregations of condensations of equal contrast. The results of numerical computations of the full mass function from the model are given in § 7. Finally, § 8 is devoted to conclusions and discussion.

2. FROM RANDOM WALKS TO BRANCHING

First we recall the reformulation in terms of random walks by Bond et al. (1991) of the phenomenological approach to the mass distribution. In such an approach, the number of condensed structures is derived from a statistical count of overdense regions in the initial density field.

Such perturbations (see Peebles 1993) have random phases and Fourier power spectrum given by $|\delta_k|^2 = Ak^{n_p}T^2(k)$, where n_p is the primordial spectral index and $T(k)$ is a transfer function depending on the nature of the matter field. The distribution of the fluctuations is taken to be a Gaussian with variance

$$S(M) = \langle |\delta(M)|^2 \rangle = \sum_k \langle |\delta_k|^2 \rangle W_M^2(k) , \qquad (2.1)$$

where $W_M(k)$ is a filtering function defining the amount of mass M to be associated with the scale corresponding to the wavelength $2\pi/k$. In a critical FRW universe, the perturbations grow as $t^{2/3}$. Thus, if the spectrum can be approximated piecewise with a power law $|\delta_k|^2 = Ak^n$, the variance takes the simple form $S(m) = m^{-2a}$ where $a \equiv (n + 3)/6$, and $m \equiv M/M_c(t)$; the characteristic mass $M_c(t) \propto t^{2/3a}$ corresponds at any given t to the scale at which $S = 1$ holds. In the following, we will use throughout the independent variable

$$\mu \equiv M^{-2a} = S(m)(t/t_0)^{-4/3} , \quad (2.2)$$

with the original meaning of resolution in mass.

Bond et al. (1991) argued that, when the perturbation field is filtered with a filtering function $W_M(k)$ sharp in the k-space, the "trajectories" $\delta(\mu)$ execute a simple random walk. Their distribution $Q(\delta, \mu)d\delta$ then obeys diffusion equation

$$\partial_\mu Q = D\, \partial_{\delta\delta} Q/2 , \quad (2.3)$$

with the starting condition at $\mu = \mu_0$ given by a narrow Gaussian. Moreover, the boundary condition of absorption at the threshold $\delta = \delta_c$ is imposed; then the appropriate starting condition becomes the sum of a symmetric narrow Gaussian centered at $\delta = 0$ and of its negative image at $\delta = 2\delta_c$. Following the latter authors, it is convenient to consider the field at the fixed time t_0 and to use a threshold $\delta_c(t) = \delta_0(t/t_0)^{-2/3}$ that decreases with t inversely to the linear growth of the perturbations. The mass function is computed on equating the fraction of mass condensed at the scale M and time t, to the rate at which the trajectories upcross the absorbing boundary placed at $\delta_c(t)$. This yields

$$N(m, t)m\, dm = N_T\, d \int_{-\infty}^{\delta_c(t)} d\delta\, Q(\delta, \mu) , \quad (2.4)$$

where $N_T(t)$ is the total number of condensations at the time t, in the present case $N_T \propto \rho/M_c(t)$; the integration extends over the range of the *surviving* trajectories. Equations (2.3) and (2.4) yield the canonical PS mass distribution: $N(m) \propto \delta_{c0}\, m^{a-2} e^{-\delta_0{}^2 m^{2a}/2}$.

In sum, the statistics of condensations from direct collapses is derived on the basis of a Markovian random walk for trajectories which have a density $Q(\delta, \mu)d\delta$, restricted by a boundary condition. Numerical realizations of a number of purely diffusing walks are visualized in Figure 2*a* of Paper I.

But dynamics requires *more* than pure diffusion, as we shall see in the following. So, taking up the developments in Paper I, we proceed to include aggregations in such framework.

Technically, first we denote by x any of the dynamical variables contained in the dynamical equations: the density contrast δ itself, the velocity potential ψ, or the gravitational potential ϕ. It will be seen that x is a *dummy* variable, since it will enter the mass distribution in integrated form.

Then we note that, due to the random initial condition, x is a *stochastic* variable fluctuating when the resolution μ increases as to describe trajectories $x(\mu)$, the counterparts of $\delta(\mu)$ defined above. To include aggregations, we allow the trajectories $x(\mu)$ also to branch as the mass resolution μ increases, corresponding to aggregations in the physical direction of increasing mass.

The overall random process will be a *branching diffusion* (BD); see Feller (1966), Karlin & Taylor (1975), and Derrida & Spohn (1988).

Such a BD model may be visualized with a Cayley tree as illustrated in Figure 1. On each path j of the tree, random increments Δ_i are extracted along the links i. Independently, the paths may also *change* in number with an elementary probability given by the function $\pi(x)d\mu$; e.g., they may branch into two, the case we shall focus on, or even die out.

For each path j, we calculate the sums $x_j = \sum_i^\mu \Delta i + \epsilon(\Delta_i)$ and $v_j = \sum_i \epsilon(\Delta_i)$, where i labels the links of the path preceding the current generation number μ and where $\epsilon_j(\Delta)$ weights the multiplicity of trajectories, being zero when no new trajectories are generated.

We will use as a basic quantity the distribution[1]

$$Z(x, \mu) = \frac{1}{\sqrt{2\pi\, d\mu_0}} \sum_{\text{all paths } j} e^{-[(x+x_j)^2 + v_j]/2d\mu_0} . \quad (2.5)$$

With such definition, the starting condition at $\mu_0 \to 0$ (that is, at $M \to \infty$ after eq. [2.2]) is the Gaussian $Z(x, \mu_0) \propto e^{-x^2/2\mu_0}/(\mu_0)^{1/2}$. This condition is set to describe the field x at such large scales that the rare perturbations are well isolated.

Note that, from the definition eq. (2.5), the distribution of x at the step μ is obtained from the initial condition by adding the quantities x_j and v_j to x. In particular, from the step μ to $\mu + d\mu$ the fluctuating distribution $Z(x, \mu)$ "evolves" following the recursion relation derived from equation (2.5):

$$Z(x, \mu + d\mu) = \begin{cases} Z(x + \Delta, \mu) & \text{with probability } 1 - \pi(x)d\mu , \\ e^{-\epsilon(\Delta)}[Z_1(x + \Delta, \mu) + Z_2(x + \Delta, \mu)] & \\ \qquad\qquad \text{with probability } \pi(x)d\mu . \end{cases} \quad (2.6)$$

Our primary concern will be the average $\langle Z \rangle$ over all the realization of the random process; this will be referred to in the following as the density of trajectories.

From equation (2.6) and for a Gaussian distribution of fluctuations Δ with variance $d\mu$, in the continuous limit $d\mu \to 0$ the average $\langle Z(x, \mu) \rangle$ is found (see Appendix A) to satisfy

$$\partial_\mu \langle Z \rangle = D\, \partial_{xx} \langle Z \rangle/2 + \eta(x)\langle Z \rangle , \quad (2.7)$$

where $\eta(x) = \pi(x)(2\langle e^{-\epsilon(\Delta)} \rangle - 1)$.

In a stochastic framework, macroscopic quantities must be related to moments of Z. The basic *macroscopic* quantity, which is finite and conserved, is constituted by the condensed mass fraction, itself the *first* moment of $N(M)$; this is equated to the *first* statistical measure of the random walk population, the average $\langle Z \rangle$, to read

$$\int_0^m dm'\, N(m')m' = N_T \int_{x_c}^\infty dx \langle Z(x, \mu) \rangle , \quad (2.8)$$

where $x_c = x_0(t/t_0)^{-2/3}$ has the same time behavior as $\delta_c(t)$. This constitutes the appropriate reformulation of the assumption (2.4) adopted by PS and by Bond et al. (1991). The system of equations (2.7) and (2.8) defines our BD *model* for the mass distribution.

We will be interested also in moments $\langle Z^m \rangle$ higher than the first. So it is convenient to introduce a generating function

$$G_q(x, \mu) \equiv \langle e^{(q-1)Z(x,\mu)} \rangle , \quad (2.9)$$

from which the mth moments of Z obtain by successive differentiations $[\partial_q^m G]_{q=1}$ with respect to the dummy vari-

[1] The function $Z(x, \mu)$ is analogous, in the plane (x, μ), to a Boltzmann energy distribution, in which the terms $(x + x_j)^2$ and v_j play the role of kinetic and potential energy, respectively. The first is related to the Gaussian fluctuation of the x field as noted in Paper I, while the second is an additional "energy" change weighing the random aggregation (represented as branching), as will be seen shortly.

able q at $q = 1$. The generating function $G_q(x, \mu)$ is governed by

$$\partial_\mu G_q = D\,\partial_{xx} G_q/2 + \eta(x)(G_q^2 - G_q)\,, \qquad (2.10)$$

an equation of the KPP type (Kolmogorov, Petrovsky, & Piskunov 1937); see Paper I. Note that equation (2.7) is recovered easily from equation (2.10) by differentiating it at $q = 1$ and using the defining equation (2.9); see also Gärtner & Molchanov (1990).

In terms of G, the relation (2.8) in turn is recovered by differentiating $[\partial_q(\cdot)]_{q=1}$ both members of the relation

$$\int^m dm'\, N(m')q^{m'} = N_T \int_{x_c}^{\infty} dx\, G_q(x, \mu)\,, \qquad (2.11)$$

which relates the Laplace (Mellin) transform of $N(M)$ to the generating function of the tree.

It is seen by comparison of equations (2.7) and (2.8) with equations (2.3) and (2.4) that, as the branching rate $\pi(x)$ vanishes, $\eta(x) \to 0$ obtains and the BD formalism goes over to the pure diffusive one of Bond et al. (1991), which yields the PS formula. The opposite limit of "frozen" fluctuations, i.e., when $p(\Delta)$ shrinks to a delta function, corresponds to a vanishing $D \to 0$ (Appendix B), and it will be shown in § 6 to yield the Smoluchowski equation.

To this point, the function $\eta(x)$ coupling diffusion and branching enters the model as a phenomenological parameter. Next we intend to discuss how the structure of equations (2.7) and (2.8), and even the function $\eta(x)$, can be *derived* from the equations governing the *dynamics* of the cosmological fluid. We shall examine in detail the case in which the adhesion approximation is used.

3. STATISTICS FROM THE ADHESION APPROXIMATION

The full description of the density and velocity fields of the cosmic fluid under gravity is of course provided by the continuity, Euler, and Poisson equations. These may be written in expanding coordinates and conformal time τ (with $\tau \sim t^{2/3}$ for the critical FRW cosmology) and can be solved in the Zeldovich approximation (Zeldovich 1970; Shandarin & Zeldovich 1989) on assuming the equality of the gravitational with the velocity potential, namely, $\phi \simeq \psi \equiv -\int^l dl'\, V$ (the velocity field has zero vorticity).

The description is still exact in one dimension up to the occurence of the multistream regions, which are generic of the highly nonlinear behavior of the original equations. In such regions, and for any dimensionality, the introduction of a vanishingly small diffusion term $\nu\nabla^2\nabla\psi$ ($\nu \to 0$) in the Euler equation allows matter to stick emulating the overall action of self-gravity (adhesion approximation, AA; Gurbatov et al. 1989). Accordingly, the velocity field V is governed by the Burgers equation

$$\partial_\tau V + (V \cdot \nabla)V = \nu\nabla^2 V\,, \qquad (3.1)$$

with the proviso that the limit $\nu \to 0$ be applied to the solutions. These have been tested by comparison with N-body simulations (see Weinberg & Gunn 1990; Kofman et al. 1992; Melott, Shandarin, & Weinberg 1994) and found to provide quite an accurate description for the grand design of the condensation process over scales larger than ~ 1 Mpc and smaller than the limit discussed by Doroshkevich et al. (1995), Shandarin (1995), and Sahni & Coles (1995). The cost of such a performance is some blurring of the condensation process; many details and most internal structures are lost into collapses delayed up to the transition to the final singularity.

The Burgers equation retains substantial nonlinear behavior, to the effect that points which are separated in the initial (Lagrangian) coordinates by an interval $\mathscr{L}$ eventually converge into formal singularities ("shocks") in Eulerian space. The mass associated with a shock will scale as $m \propto \mathscr{L}$ in unidimensional motion and as $\mathscr{L}^d$ for isotropic shocks in a space with dimensionality d.

The solutions of the Burgers equation are amenable to analytic expression, or to geometric representations, in terms of the *initial* conditions (Gurbatov et al. 1989). Stochastic initial conditions for the density, with power spectral index n as stated, translate into related stochasticity for the velocity or the gravitational potential, after the Fourier transform of the Poisson equation $|\psi_k|^2 \propto k^{-4}|\delta_k|^2$. Such initial conditions, together with the three original equations, define the coalescence length $L_c(t) \propto t^{2/(n+d)}$ [corresponding to $M_c(t)$], such that at time t the distribution of perturbations on scales $\mathscr{L} \gg L_c(t)$ is still essentially unconstrained. One asks for the statistical distribution of the masses within the shocks developed at a given time t.

Sinai (1992a) provided, under very specific conditions (one-dimensional, Brownian motion ($d = 1$, $n = 0$), and small masses [$l = \mathscr{L}/L_c(t) \ll 1$]), a neat geometrical construction reducing the answer to a simple statistics concerning the *initial* ψ. This expresses the "isolation" condition for the shock on a given scale not to be buried within larger ones; the initial potential is required to be sufficiently shallow as to satisfy a non–shock condition on progressively smaller scales down to that under consideration. The condition is phrased in terms of the probability

$$p_\mu = \text{prob}\,\{\psi_i > -x_c,\ i = 1, \ldots, \mu\}\,, \qquad (3.2)$$

where $x_c \sim 1$, and $\mu(M)$ has the *same* scaling (eq. [2.2]) as the step number in the BD model introduced in § 2; accordingly, we will use the same symbol for both. Vergassola et al. (1994) stress that the condition (3.2), derived as a non–crossing condition for the dynamical streams, corresponds to a non–collapse condition in the language of density contrasts.

Since the ψ values are integrals, they may be viewed as partial sums $\psi_i = \sum_{j=1}^{i} x_j$ of random variables x_j, and sothe condition (3.2) fits into the classical scheme of a symmetric walk (of the general Levy kind; see Feller 1966) starting from a value x_*, ending with x_{**}, and staying on a definite side of $x_c = x_* + x_{**}$ for at least $\mu = O(m^{-2a})$ steps. In the limit of small m, Sinai (1992a) proves the scaling of the mass function to be given by the relation $\int^m dm'\, N(m')m' \propto p_\mu$.

Vergassola et al. (1994) recently extended such statistics to initial conditions with $n \neq 0$ and emphasized its analytical description in terms of a recursive relation for the density $P_\mu(x)$, whose integral $p_\mu = \int_{-x_c}^{\infty} dx\, P_\mu(x)$ measures the *surviving* walk population.

We write the recursive relation and the expression for $N(m)$ in the form

$$P_{\mu+d\mu}(x) = \int_{-x_c}^{\infty} dy\, \kappa(x-y)P_\mu(y) \quad P_{\mu 0}(x) = \kappa(x) \text{ even}\,, \qquad (3.3)$$

$$N(m)m\, dm = N_T\, dp_\mu = N_T\, d\int_{-x_c}^{\infty} dx\, P_\mu(x)\,, \qquad (3.4)$$

where $x_c = x_0(t/t_0)^{-2/3}$. Vergassola et al. (1994) set $x_c = 0$, but for reasons to become clear in § 5 we have reatained for the "isolation" condition the original form prob $\{\sum_{j=1}^{i} x_j > -(x_* + x_{**}),\ i = 1, \ldots, \mu\}$. Correspondingly, we retain the threshold $x_c = x_* + x_{**}$. For the sake of definiteness, we shall refer to the specific, but not mandatory, Gaussian form of the kernel provided by $\kappa(x) = e^{-x^2/2D\,d\mu}/(2\pi D\,d\mu)^{1/2}$.

The *similarity* of equation (3.4), relating $N(m)$ to $P_\mu(x)$, with equation (2.8), relating $N(m)$ to the average density $\langle Z\rangle$ of diffusing trajectories in the BD process, is apparent. The key quantity to yield the mass fraction condensed in dm is given in equation (3.4), quite as in equation (2.8), by the differential variation with μ of p_μ, the partial area under $P_\mu(x)$ to the right of $-x_c$, which measures the surviving trajectory population. Moreover, we will show that equation (3.3) constitutes a close *counterpart* of equation (2.7), with the threshold $-x_c$ appearing explicitly as an integration limit in the integral (3.3) to express the one-sidedness condition for ψ.

In fact, we intended to show that the model provided by equations (2.7) and (2.8) converges with the results from the above equations (3.3) and (3.4) in the regime $d = 1$, $m \ll 1$ where the latter had been originally derived from the AA. In addition, we will show that such *equivalence* extends to isotropic three-dimensional condensations and to the range $m \gg 1$.

To these aims, we note first that the proper extension of equations (3.3) and (3.4) to $d = 3$ obtains, differently from Vergassola et al. (1994), on considering only *isotropic* singularities of the node kind, as opposed to walls and filaments. The former correspond better to groups or clusters of galaxies identified in the real sky or in N-body simulations. It is the counting of such singularities that may be expected to scale like the PS formula. For isotropic condensations, $m \propto l^3$ holds, and meanwhile in the regime $l < 1$ their probability scales as $P_{l^3} \propto P_l^3$. So a power law $N(M)$, as we will show to be the case for small m, will retain its scaling in passing from $d = 1$ to $d = 3$. At the large-mass end, isotropic shocks will dominate anyway.

With these provisos, we proceed to discuss, in the small-mass regime $m < 1$ to begin with, the equivalence of a BD model of the kind proposed in § 2 with the outcomes from AA dynamics.

4. SMALL MASSES

Since $\mu \propto S \propto m^{-2a}$ holds, the small-mass regime corresponds to large values of μ. We begin by writing equation (3.2) in the form

$$P_{\mu+d\mu}(x) = \left[\int_{-\infty}^{\infty} dy\,\kappa(x-y) - \int_{-\infty}^{-x_c} dy\,\kappa(x-y)\right] P_\mu(y)\,, \tag{4.1}$$

where the first integral is fully symmetrical. If the functions $P_\mu(x)$ were evolved after such a symmetrical integral operator alone, they would be simple Gaussians $P_\mu(x) = e^{-x^2/2D\mu}/(2\pi D\mu)^{1/2}$ with variance growing by progressive summation of the basic variances $D\,d\mu$ in the kernel $\kappa(x)$. Then the total area under $P_\mu(x)$ would be conserved, and the median value of x would be zero. In such a case, it is easily checked by direct integration that the partial area satisfies asymptotically the canonical scaling $p_\mu \to \mu^{-1/2}$. Such behavior yields the small-mass PS scaling, as follows easily by using equation (3.4).

Now we recast equation (4.1) into an equivalent *differential* form. For that, the first and fully symmetric integral may be developed for $d\mu \to 0$, when considering the small variance $D\,d\mu$ of the starting condition $P_{\mu 0}(x)$, and the even symmetry of $\kappa(x)$ which cancels on average the first derivative of $P_\mu(x)$. The result is

$$\int_{-\infty}^{\infty} dy\,\kappa(y) P_\mu(x-y) \simeq P_\mu(x) + D\,d\mu\,\partial_{xx} P_\mu/2\,. \tag{4.2}$$

On the other hand, the left-hand side of equation (4.1) in the same limit writes $P_{\mu+d\mu} \simeq P_\mu + d\mu\,\partial_\mu P_\mu$. Equating these two expressions, one would obtain the simple diffusion equation $\partial_\mu P_\mu = D\,\partial_{xx} P_\mu/2$, whose Green function is of course a spreading Gaussian yielding the same results as in the previous paragraph.

But equation (4.1) includes a second, negative term. This latter integral may be represented, to the leading order when the scale for $P_\mu(x)$ to vary is much larger than the scale of the kernel $\kappa(y)$, with the simple product $\eta_A(x)P_\mu(x)$, where

$$\eta_A(x) = -\int_{-\infty}^{-x_c-x} dy\,\kappa(y) \tag{4.3}$$

has a steep, stepwise drop for $x > -x_c$. The result is now

$$\partial_\mu P_\mu = D\,\partial_{xx} P_\mu/2 + \eta_A(x) P_\mu\,. \tag{4.4}$$

The latter equation can be fit into the pattern provided by the BD equation (2.7) on identifying $\langle Z(x, \mu)\rangle$ with $P_\mu(x)$ and η with η_A. Then equation (4.3) provides the prescription

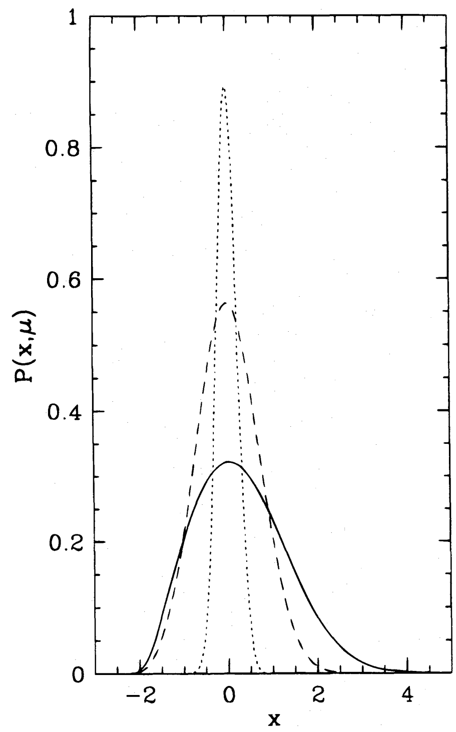

FIG. 2.—The solution $P_\mu(x)$ of eq. (3.3) computed numerically for $x_0 = 1.69$ at various μ, from the starting condition μ_0 (*dotted line, scaled down by a factor 1/2 for graphic reasons*) to $\mu = 30\mu_0$ (*solid line*). Note how the spread from the starting condition and the loss of area for $x < -x_0$ combine to lead to a strong deviation from the starting Gaussian. Nearly identical results obtain from eq. (2.7) with the branching parameter $\eta_A(x)$ given by eq. (4.3).

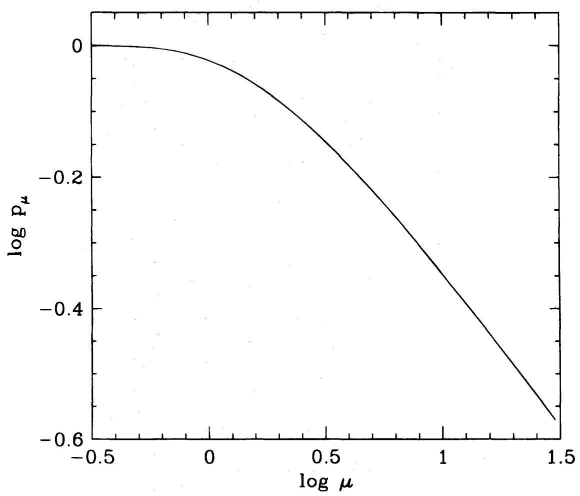

FIG. 3.—The evolution with increasing μ of the area $p_\mu = \int_{-x_c}^{\infty} dx\, P_\mu(x)$. Note the universal behavior $p_\mu \to \mu^{-1/2}$ at large μ.

from AA for the combination of the functions $\pi(x)$ and $\epsilon(\Delta)$ entering $\eta(x)$ in the BD model.

This proves the *equivalence* in the small-mass limit of the statistics from AA and the BD model, with the former providing the specific function $\eta(x)$. The starting condition for $P_{\mu 0}(x)$ can be taken to be a Gaussian just as $\langle Z_0(x)\rangle$. The PS-like power law is recovered in this limit, as illustrated by Figures 2, 3, and 4.

The origin of such behavior can be understood from the numerical solution of equations (3.3) or (4.4); see Figure 2. The "evolution" of the surviving trajectory distribution $P_\mu(x)$ may be described as a spread of $P_\mu(x)$ from the narrow starting condition, combined with loss of the underlying area caused by the term $\eta_A(x)P(x)$ at negative values of x up to the boundary $-x_c$ and slightly beyond. The function $P_\mu(x)$ is depressed for values $x < -x_c$ where it vanishes for increasing μ, while its maximum shifts to the right, and the right tail is relatively enhanced. This reflects the circumstance that in our random process the median value of those trajectories that never reached the boundary within μ steps is to increase with μ itself, since only the distant ones will survive. The effect of the stepwise shape of $\eta_A(x)$ is equivalent to a boundary condition of the absorbing wall type, namely, $P_\mu(-x_c) = 0$. The loss at the boundary will decrease the total number of surviving walks, corresponding to the total area $\int_{-\infty}^{\infty} dx\, P_\mu(x)$; Figure 3 shows that this asymptotes to $\mu^{-1/2}$. But since essentially no area remains to the left of $-x_c$, the same behavior is eventually shared by p_μ, complying with the "universal asymptotic property" of symmetric walks (Feller 1966). This in turn yields the power-law scaling of $N(m)$ at small masses (See Fig. 4), which converges to the PS scaling.

We proceed now to examine how we can extend the equivalence beyond the range $m \ll 1$ for which the results of Sinai (1992a) and Vergassola et al. (1994) were derived originally.

5. LARGE MASSES

Large masses $m > 1$ correspond to small values of μ. In the specific BD model with $\eta(x) = \eta_A(x)$, there will be a range of μ close to the starting condition at μ_0, when effectively $\eta(x)\langle Z_\mu(x)\rangle \simeq 0$ holds for the relevant range of $x \simeq 0$.

Formally, this holds because the starting condition is a narrow Gaussian $Z_0(x) \propto e^{-x^2/2D\mu_0}/(\mu_0)^{1/2}$ with vanishing μ_0. So $\langle Z_\mu(x)\rangle$ will differ appreciably from zero only for small values of x, where $\eta_A(x)$ is small.

Physically, at large scales the density field is essentially unprocessed. Large mass condensations are so rare that they have little chance of being comprised within larger ones, or of interacting. In such conditions, the model equation (2.7) yields

$$\partial_\mu\langle Z\rangle \simeq D\,\partial_{xx}\langle Z\rangle/2\,. \tag{5.1}$$

The solution of equation (5.1) is the spreading Gaussian $\langle Z(x, \mu)\rangle = e^{-x^2/2D\mu}/(2\pi D\mu)^{1/2}$. Use of the rule in equation (2.8) now yields for $m \gg 1$ the PS cutoff

$$\ln N(m) \propto -x_0^2 m^{2a}/2\,, \tag{5.2}$$

as a direct reflection of the Gaussian starting condition.

On the other hand, Vergassola et al. (1994) expected from the AA dynamics on heuristic grounds that $\ln N(m) \propto -m^{2a}$ should hold for $m > 1$. Thus, in the large-mass limit as well, the BD model is equivalent to the statistics from AA.

We add that such behavior of $N(m)$ is formally retained in equations (3.3) and (3.4), although they were originally derived only for the small-mass regime, *provided* that $x_c \neq 0$. Then for small values of $\mu - \mu_0$ the inequality $\mu - \mu_0$ the inequality $\mu - \mu_0 \ll x_c^2$ is bound to hold. As long as $P_\mu(x)$, like the starting $P_{\mu 0}(x)$, is still so narrow as to decay toward the left on a scale much smaller than x_c, the lower integration limit in equation (3.3) may be replaced by $-\infty$, and the equation written

$$P_{\mu+d\mu}(x) \simeq \int_{-\infty}^{\infty} dy\, \kappa(x-y)P_\mu(y)\,. \tag{5.3}$$

This is just another representation of the freely diffusing Gaussian, with consistently steep decay.

So the analytical expressions (3.3) and (3.4) for the shock statistics from AA yield the correct $N(m)$ even in the range $m > 1$. The same result is provided by the BD model with the specific value of $\eta(x) = \eta_A(x)$ given by equation (4.3). From this and the previous section, we conclude that the model and the analytical formulation of the statistics from the AA converge for both $m < 1$ and $m > 1$.

Having so *calibrated* the model at the opposite extremes, we proceed to examine its content.

6. AGGREGATIONS

The generic BD model comprises binary aggregations of condensations, which we are going to single out under special conditions and for the specific model form derived from AA.

We focus on condensations of *equal* density contrasts $x = \bar{x}$, so that $G_p(x, \mu)$ is peaked at $\bar{x}$ and equation (2.11) reads

$$\begin{aligned}\int^{m'} N(m')q^{m'}\,dm' &= N_T \int_{x_c}^{\infty} dx\, G_q(x, \mu)\delta_D(x-\bar{x})\\ &= N_T\, G_q(\bar{x}, \mu)\,,\end{aligned} \tag{6.1}$$

where $\delta_D(x)$ is the Dirac delta function. Expressing the first integral in equation (6.1) as a finite sum, and substituting the definition in equation (2.9) of the generating function $G_q(x, \mu)$, we find

$$\sum_m q^m N(m) = N_T\langle e^{(q-1)Z(\bar{x},\mu)}\rangle = \sum_m \frac{(q-1)^m}{m!}\langle Z^m\rangle\,. \tag{6.2}$$

Applying the operator $[\partial_q^m(\cdot)]_{q=1}$ to both members, we find

$$\frac{N(m)}{N_T} = \frac{\langle Z^m(\bar{x}, \mu)\rangle}{m!} . \quad (6.3)$$

The rate of change of the right-hand side can be derived from the recursion relation (2.6) in the case $x = \bar{x}$, when the fluctuating component Δ of the random walk is frozen; then the probability distribution $p(\Delta)$ is shrunken to a delta function, and correspondingly the branching rate is fixed to the constant value $\pi(\bar{x})$. In this condition from equation (2.6), it follows that (see Appendix B)

$$\partial_\mu \frac{\langle Z^m\rangle_\mu}{m!} = \pi(\bar{x}) \sum_{k=1}^{m} \frac{\langle Z^k\rangle_\mu}{k!} \frac{\langle Z^{m-k}\rangle_\mu}{(m-k)!} - \pi(\bar{x}) \frac{\langle Z^m\rangle_\mu}{m!} . \quad (6.4)$$

Note that the quadratic term arises from the binomial expansion (shown by eqs. [B1] and [B2]) of the mth moments of Z appearing in equations (6.3).

In our limit of frozen fluctuations $\langle\delta^2\rangle \propto S =$ const. holds, and equation (2.2) yields $\mu \propto t^{-4/3}$. Using this relation and equation (6.3), we obtain from equation (6.4), as in Paper I, the discretized form of the Smoluchowski aggregation equation for $N(m)$:

$$\partial_t N = \tfrac{1}{2} \int_0^m dm' \, K(t) N(m', t) N(m - m', t) - N(m, t) \int_0^\infty dm' \, K(t) N(m', t) . \quad (6.5)$$

Here the total number of aggregates $N_T = \int dm \, N(m)$ decreases with time according to $d\mu/dt = -(dN_T/dt)/N_T$, and the kernel $K(t) = \pi(\bar{x})(2/N_T)(d\mu/dt)$ is a decreasing function of time here independent of M. The above derivation holds only for condensations with equal contrast; the complete case is treated numerically in § 7.

In the specific application to the statistics derived from AA, we show in Appendix B that $\pi(x) \propto |\eta_A(x)|$ holds so that the aggregation kernel K becomes rapidly small to the right of $-x_c$. Our interpretation of the term $\eta_A(x)P(x)$ in equation (4.4) as a trace branching component still discernible in AA is not mandatory, but it is consistent with equation (2.7) governing the average density $\langle Z\rangle$ of trajectories, which determines $N(m)$. Moreover, such an interpretation establishes a useful link with more complex dynamics than AA, which will be discussed further in § 8.

7. NUMERICAL SOLUTIONS OVER THE FULL RANGE

The distribution over the full mass range is shown in Figure 4 as computed numerically from the system of equations (3.3) and (3.4), equivalent to the complete BD model including both Guassian fluctuations and branching.

The *only* free parameters is the threshold x_0. For a fair comparison with the PS formula, we set $x_0 = 1.69$, the canonical value for δ_c, and we superpose the two plots at the low-mass end. We note the following features:

1. While the distribution $N(m)$ we compute has the same behavior as the PS formula at the two ends (see §§ 4 and 5), its average gradient is steeper due to the intermediate range $m \sim 1$, where the interpretation of § 6 applies. A steeper gradient is also consistent with the high resolution of the shocks implied by AA, which separates condensations often unresolved by the filtering procedure.

2. The normalizations necessarily differ. The result from BD for isotropic condensations will have a lower normalization since much mass, and in fact most mass at the low end (see Vergassola et al. 1994), resides in shocks with topology different from isotropic, namely, of the wall and filament kinds.

3. It is not necessary, and in fact not even likely, that $x_0 = \delta_c = 1.69$ should hold; certainly x_0 is not fixed only by the dynamics of a single collapse pattern, such as the top-hat one.[2] However, by assuming the equality, the remaining difference with the PS formula must be traced back to the extra term $\eta\langle Z\rangle$ in equation (2.7) as compared with equation (2.3); this is responsible for the above features.

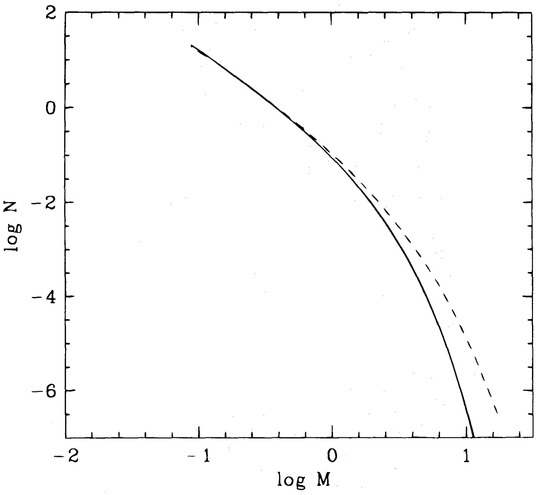

FIG. 4.—The mass distribution $N(M)$ (*solid line*) computed numerically from the BD model defined by eqs. (2.7) and (2.8), with the branching parameter $\eta_A(x)$ given by eq. (4.3). The result is compared with the PS mass distribution (*dotted line*) with the same value of the threshold $\delta_c = x_0 = 1.69$.

8. SUMMARY AND DISCUSSION

This work addressed the theory of the mass distribution $N(M, t)$. It is focused on the *foundations* provided by the basic dynamical equations for the cosmic fluid under gravity.

Our main point is that, even treating the dynamics in the simplified form of the adhesion approximation, $N(M, t)$ arises from a stochastic process *more* complex than the diffusive random walk yielding the PS distribution. In fact, the process comprises *changes* in the number of the walks, as described in the simplest form by a linear term added to the diffusion equation.

Our bottom line is that the statistics of physical condensations can be reduced to counting representative trajectories in a stochastic process of the *branching diffusion* kind, represented by a Cayley tree. We have gone through the following steps.

8.1. *From Dynamics to Press & Schechter and Smoluchowski*

We took advantage of the convenient recasting of the dynamics in Lagrangian terms provided by the *adhesion approximation* (AA, § 3), which retains substantial nonlinearity yet is amenable to explicit solutions in terms of initial conditions. This reproduces the grand design of the struc-

[2] The quantities x_* and x_{**} added to make x_c are random variables of order 1. Their distribution is different from that of x and is being studied by Sinai (1992b).

tures and (within the limits discussed by Doroshkevich et al. 1995, Shandarin 1995, and Sahni & Coles 1995) the number, the places, and the masses of the fully developed condensations, but it represents them as singularities bypassing many details of their formation and most of their internal structure. Within AA, recent work (Sinai 1992a; Vergassola et al. 1994) has shown, under particular conditions, how to derive the asymptotic small-mass distribution of such singularities. This is reduced to a statistics over the initial potential perturbations.

In this paper we show that such statistics can be formulated effectively in terms of a *branching diffusion* process, that is, a random process as considered in Paper I which includes not only Gaussian fluctuations as the mass resolution μ increases, but also change in the trajectory number by branching. These features may be visualized (see Fig. 1) by a *Cayley tree.*

The computation of the condensation number at any given scale is then reduced to counting the representative trajectories found in such a random process at the corresponding mass resolution μ as given by equation (2.8). The computation may be performed either numerically using the Cayley tree, or analytically from a *modified diffusion* equation like equation (2.7).

The AA provides a specific form for the *branching* rate entering the above equation. The PS mass distribution or the Smoluchowski aggregation equation are derived as special cases by restricting the counts to pure *diffusion*, corresponding to straight collapses from the initial perturbations, or to pure *branching*, corresponding to aggregation of equal-contrast condensations.

Thus, a bridging is established between the dynamics of the cosmic matter field and the two currently used phenomenological mass distributions, and a dynamical foundation is provided to the model proposed in Paper I.

8.2. *Structure of N(M)*

The *specific* BD model calibrated with the AA statistics yields the mass distribution shown in Figure 4.

The high-end $m \gg 1$ corresponds (see § 5) to the collapse of rare, *isolated* perturbations. The outcome is the PS-like exponential shape for $N(m)$, directly reflecting the original Gaussian distribution assumed for such perturbations at very large scales. Indeed, any continuous process whose probability distribution is governed by a differential equation like equation (2.7) must remain for some steps close to its starting condition. If this is assumed to be a Gaussian, the results is inevitably the PS-like exponential cutoff. The actual extent of the range in which this occurs is fixed by the threshold x_0, the parameter entering the isolation condition (3.2) for the potential perturbations. With x_0 increasing, isolation holds for smaller and smaller scales, and the exponential shape extends toward smaller masses. Here natural values $x_0 \sim 1$ have been considered.

With such a choice, at the low-end $m \ll 1$ (large μ in the tree) the density of surviving trajectories is governed by a diffusion process that is *perturbed* by the term $\eta(x)\langle Z(x, \mu)\rangle$ in equation (2.7), where $\eta = \eta_A$ is provided by AA through equation (4.3). The sharp structure of $\eta_A(x)$ plays a steering role: because this function is large to the left of $-x_c$, it causes $\langle Z(x, \mu)\rangle$ to shift toward larger x as μ increases (see Fig. 2); because it becomes small toward the right, there the product $\eta(x)\langle Z(x, \mu)\rangle$ becomes negligible. The stepwise transition in $\eta_A(x)$ reflects the sudden transition from condensations to singularities in AA, and it enforces a simple scheme at low masses: the diffusive component in equation (2.7) dominates, but at the threshold the equivalent of an absorption boundary condition is established. This in turn yields $p_\mu \to \mu^{-1/2}$ (see Fig. 3) as a particular instance of the universal asymptotic property of symmetric random walks; see Feller (1966). In terms of m, the PS-like power law is recovered. The surprising persistence of the PS formula is traced back to general properties of (nonuniformly) perturbed diffusion processes.

The range $m \sim 1$ corresponds to intermediate values of μ (see Fig. 2) at which the product $\eta(x)\langle Z(x)\rangle$ is small but not negligible. There, in conditions not diluted by new collapses, one may single out (see § 6) a stochastic component corresponding to pure branching, and hence to the *aggregations* handled in detail by the Smoluchowski equation. In this range, $N(m)$ from the specific BD model is systematically steeper than the PS formula, when the comparison is performed at equal values of the parameters x_0 and δ_c.

8.3. *The Role of Aggregations*

That the term $\eta\langle Z\rangle$ just discussed may be interpreted in terms of interactions between condensations is shown in § 6. Actually, it describes those *residual* "interactions" still retained in AA. These include exclusion of small-scale perturbations from the volumes occupied by the larger ones (cf., Bond & Myers 1993) and also aggregations of isotropic shocks in the stage subsequent to their formation.

As for the aggregations, in the AA context the binary rate $\propto N^2$ is strongly suppressed by the general cosmic expansion built into the independent variables of the Burgers equation (3.1). In addition, metastable and nonexpanding structures like clusters or groups in which aggregation is most effective are reduced to singularities, so that most aggregations taking place in their interior are simply canceled. The overall ineffectiveness of aggregations in the context of AA translates into the decreasing kernel $K(t)$ for such residual aggregations appearing in the Smoluchowski equation (6.5), with the cosmological expansion entering through $d\mu/dt$.

The aggregation component finds its full scope within volumes, like groups and clusters, whose substructure is unresolved in the context of AA. The role of aggregations in richer dynamical frameworks is discussed in § 8.5.

8.4. *Discussion*

The mass distribution we derive results from counts limited to isotropic, node-type singularities within the AA. On the one hand, these are the ones most easily singled out as groups and clusters in the observations. On the other hand, isotropic condensations correspond to the isolated configurations considered in the peaks approach (see Bardeen et al. 1986; Bond & Myers 1993) with results similar to ours. We understand the similarity by interpreting (see § 2) our stochastic amplitude x as proportional to $\delta \propto E$, the perturbation energy which is the relevant, conserved variable in the case of isotropic condensations over an FRW background. For small anisotropy, the changes in E are moderate (Bond & Myers 1993).

To this point, we have neglected correlations among the various scales, but we note several circumstances making this tolerable. The scaling of the independent variable $\mu \propto m^{-2a}$ still holds for any relevant index n of the perturbation power spectrum (Kahane 1985). For large μ, the effect of initial correlations on the heavily processed $\langle Z(x, \mu)\rangle$ distributions tends to be minor; see Peacock & Heavens (1990) and Vergassola et al. (1994). Power spectra with increasing n tend to decrease the range of the correlations (Peacock & Heavens 1990). Finally, aimed and extended numerical simulations by Vergassola et al. (1994) yielded a scaling of $N(m)$ at small m consistent with the results derived from neglecting correlations. Further investigation is warranted by the importance of the issue in point of principle.

8.5. *Prospects*

A general conclusion from the above is that statistics of the gravity-dominated cosmic dynamics is feasible; in fact, in the case discussed here the derivation of the mass distribution has been reduced to a *statistics* over the *initial* conditions.

The role of the initial conditions is highlighted in the context of AA as the gravitational dynamics can be expressed explicitly as an operator (solution of Burgers equation) acting on these. Such an operator is nonlinear but fully deterministic; it is reducible to a constraint, in the form of the isolation condition (3.2) selecting particular configurations in the initial field. It is noteworthy how close this is to the exclusion procedure applied by Bond & Myers (1993) to the initial conditions of their simulations.

The statistics of the masses is expressed by equation (2.8). This simple relation connects the condensed mass fraction [the basic macroscopic quantity which is finite and conserved and is given by the first moment of $N(m)$] with the first measure of an underlying stochastic process, i.e., to the number p_μ of the corresponding *trajectories*. We stress the invariance of the results: the proper scale variable is always given by $\mu \propto m^{-2a}$; the dummy stochastic amplitude x may be interpreted in terms of different dynamical variables (δ or ϕ); different counters of the kind of $\langle Z(x, \mu)\rangle$ may be used for computing p_μ.

The basic probability p_μ, in turn, can be obtained from several equivalent schemes, which we used as convenient throughout Papers I and II:

1. The handy differential representation in terms of a perturbed diffusion equation for the trajectory density $\langle Z(x, \mu)\rangle$. The integral $\int dx \langle Z(x, \mu)\rangle$ is just p_μ.
2. The more powerful integral representation in the form of a recursive equation of the type of equation (3.3) for the trajectory density. This incorporates the constraint set by the threshold x_c, and in principle it can include correlations and richer dynamics.
3. The Cayley tree representation, with the associated computational scheme, for fluctuating/branching trajectories. In the case of no branching, this can handle correlations in the trajectories through the appropriate Langevin equations; see Bond et al. (1991), Porciani et al. (1995).

The challenge to further research is to determine what happens in going beyond the AA description of the dynamics. A pressing issue concerns any enhanced aggregation of knots in the stage subsequent to their formation, driven by "random Zeldovich motions" (see Doroshkevich et al. 1995; Shandarin 1995) of the embedding larger scale structure. The guideline we have proposed in this paper comes down to viewing the differential equation (2.7) as a continuity equation for $\langle Z(x, \mu)\rangle$ of the general form

$$\partial_\mu \langle Z\rangle + \nabla \cdot \boldsymbol{J} = S . \tag{8.1}$$

Here we have used the form for one variable x and with the lowest order approximations: for the flux, the diffusive symmetric expansion $\boldsymbol{J} = -D\, \partial_x \langle Z\rangle/2$, and for the source term the linear form $S = \eta(x)\langle Z\rangle$. These are enough to show *nonconservation* of trajectories as a generic feature of the relevant random process.

While we have just seen that AA yields a trajectory number change concentrated near the threshold, we expect that richer dynamics will provide a smoother shape of $\eta(x)$ and cause a larger proliferation of trajectories. The appropriate tool for their inclusion is constituted by the integral counterpart of the above continuity equation in the form of equation (3.3). This may include a non-Gaussian component of the kernel $\kappa(x)$ related specifically to the random motions of isotropic condensations induced by potential fluctuations on larger scales, which will lead to an additional source term S. Work along this line will be reported elsewhere.

It is a pleasure to thank B. T. Jones for stimulating us to pursue the connection of statistics with dynamics, P. Monaco for fruitful discussion and the referee for comments helping to improve our presentation. We acknowledge grants from MURST and ASI.

APPENDIX A

The probability distribution for the fluctuations Δ is

$$p(\Delta) = \frac{1}{\sqrt{2\pi D\, d\mu}}\, e^{-\Delta^2/2D d\mu} . \tag{A1}$$

We now compute the average $\langle Z\rangle = \int dZ\, \mathscr{P}(Z, \mu)$, where $\mathscr{P}(Z, \mu)$ is the probability distribution of Z at the step number μ. From the recursion relations (2.6), we find

$$\begin{aligned}\langle Z(x, \mu + d\mu)\rangle = [1 - \pi(x)d\mu] &\iint d\Delta\, p(\Delta) dZ\, \mathscr{P}(Z, \mu) Z(x + \Delta, \mu)\\ + \pi(x)d\mu &\iint d\Delta\, p(\Delta) e^{-\epsilon(\Delta)} dZ\, \mathscr{P}(Z, \mu) Z(x + \Delta, \mu)\\ + \pi(x)d\mu &\iint d\Delta\, p(\Delta) e^{-\epsilon(\Delta)} dZ\, \mathscr{P}(Z, \mu) Z(x + \Delta, \mu) ,\end{aligned} \tag{A2}$$

where the labels 1, 2, to Z have been dropped as no longer relevant in the two separated integrals. Expanding $Z(x + \Delta, \mu)$ in Taylor series for small Δ, the contribution of the first-order term $\partial_x Z(x, \mu)\Delta$ to the integrals over $d\Delta$ on the right-hand side vanishes by symmetry of $p(\Delta)$. The integrals over $d\Delta$ of the second-order term $\partial_{xx} Z(x, \mu)\Delta^2/2$ reduce to $\partial_{xx} Z(x, \mu)D\, d\mu$. Substituting into equation (A2) and keeping terms up to $O(d\mu)$, we find

$$\langle Z(x, \mu + d\mu)\rangle - \langle Z(x, \mu)\rangle = D\,\partial_{xx}\langle Z(x, \mu)\rangle d\mu/2 + \pi(x)[2\langle e^{-\epsilon(\Delta)}\rangle - 1]d\mu\langle Z(x, \mu)\rangle \,. \quad \text{(A3)}$$

Dividing by $d\mu$ and taking the limit $d\mu \to 0$, equation (2.7) is obtained.

APPENDIX B

The limit of frozen fluctuations (when the only stochastic process is branching) corresponds to a random process in which the displacements Δ are extracted from a probability distribution $p(\Delta)$ which shrinks to a Dirac delta function. As seen from equation (A1), this in turn corresponds to the limit $D \to 0$. In equation (A3), and hence in equations (2.7) and (2.10), such a limit causes the diffusion term to vanish.

With the above restriction, x is frozen to a fixed value $\bar{x}$. Choosing $\epsilon(\Delta)$ so as to give $\langle e^{-\epsilon(\Delta)}\rangle < \frac{1}{2}$, the definition of $\eta(x)$ just after equation (2.7) yields for the BD model with $\eta = \eta_A$ the relation $\pi(\bar{x}) \propto -\eta_A(\bar{x})$; note that $\eta_A < 0$ after equation (4.3). Then the recursion equation (2.6) yields the following equation for the moments of the average trajectory density:

$$\langle Z^m\rangle_{\mu+d\mu} = \pi(\bar{x})d\mu \iint dZ_1\, dZ_2\, \mathscr{P}(Z_1)\mathscr{P}(Z_2)(Z_{1\mu} + Z_{2\mu})^m + [1 - d\mu\, \pi(\bar{x})] \int dZ\, \mathscr{P}(Z)Z_\mu^m \,, \quad \text{(B1)}$$

where $\mathscr{P}(Z, \mu)$ is the probability distribution of Z at the step number μ. Expanding the binomial, this becomes

$$\langle Z^m\rangle_{\mu+d\mu} = d\mu\, \pi(\bar{x}) \iint dZ_1\, dZ_2\, \mathscr{P}(Z_1)\mathscr{P}(Z_2) \sum_{k=1}^{m} \frac{m!}{k!(m-k)!} Z_{1\mu}^k Z_{2\mu}^{m-k} + [1 - d\mu\, \pi(\bar{x})]\langle Z^m\rangle_\mu \,. \quad \text{(B2)}$$

In the continuous limit $d\mu \to 0$, we find

$$\partial_\mu \frac{\langle Z^m\rangle_\mu}{m!} = \pi(\bar{x}) \sum_{k=1}^{m} \frac{\langle Z^k\rangle_\mu}{k!} \frac{\langle Z^{m-k}\rangle_\mu}{(m-k)!} - \pi(\bar{x}) \frac{\langle Z^m\rangle_\mu}{m!} \,, \quad \text{(B3)}$$

which is equation (6.4).

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